A RIGOROUS PROOF OF $P \neq NP$ VIA REFUTATION OF THE COMPRESSIBILITY HYPOTHESIS FOR THE SOLUTION SPACE OF SAT

by

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Abstract. — The question of whether $P = NP$ remains a central challenge in theoretical computer science and modern mathematics. This paper presents a rigorous deterministic proof that $P \neq NP$. The proof is based on refuting the Compressibility Hypothesis (CH) for the Boolean satisfiability problem (SAT), a canonical NP-complete problem. We demonstrate that no polynomial-time algorithm can universally compress the solution space of SAT to guarantee the presence of at least one solution in a polynomial-sized set. This refutation is achieved by constructing a special self-referential formula φ_f for any assumed compression function f, showing that the formula is satisfiable, but the set of assignments produced by f for φ_f contains none of its solutions. The resulting contradiction with the definition of f refutes CH and proves $P \neq NP$.

1. Introduction

The complexity classes P and NP, introduced in the early 1970s, categorize computational tasks based on their complexity on deterministic and nondeterministic Turing machines. The class P includes tasks solvable by a deterministic machine in polynomial time. The class NP includes tasks for which a proposed solution can be verified by a deterministic machine in polynomial time. Equivalently, NP is the class of tasks solvable by a nondeterministic machine in polynomial time.

The question $P = NP$ is one of the seven Millennium Prize Problems posed by the Clay Mathematics Institute, and its resolution has profound implications for science and technology. If $P = NP$, many tasks currently considered practically unsolvable (e.g., the traveling salesman problem, optimization problems in logistics, financial modeling, or cryptanalysis) would become efficiently solvable.

NP-complete problems, introduced by Stephen Cook and Leonid Levin, are the hardest problems in NP, in the sense that any problem in NP can be reduced to an NP-complete problem in polynomial time. The Boolean satisfiability problem (SAT) was the first problem proven to be NP-complete. Consequently, $P = NP$ if and only if SAT $\in P$.

The intuitive asymmetry between finding a solution for SAT (which appears exponential) and verifying a proposed solution (polynomial) underpins the conjecture $P \neq NP$. This paper formalizes this intuition by relating the possibility of fast solution search to the ability to "compress" the search space.

We present a proof of $P \neq NP$ by refuting the Compressibility Hypothesis (CH) for SAT. CH posits the existence of a polynomial-time algorithm that produces a polynomial-sized set of assignments, which, for any satisfiable SAT formula, is guaranteed to contain

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at least one solution. We will prove that such a compression function does not exist, using a diagonalization construction of a self-referential formula that serves as a counterexample to any assumed compression function.

2. Main Definitions

We use standard definitions from computational complexity theory.

***Definition 2.1* (Class P).** — The class P is the set of decision problems solvable by a deterministic Turing machine in polynomial time.

***Definition 2.2* (Class NP).** — The class NP is the set of decision problems L for which there exists a polynomially bounded verifier $V(x, y)$, running in polynomial time with respect to $|x| + |y|$, such that $x \in L \iff \exists y \in \{0,1\}^{\text{poly}(|x|)}$ with $V(x, y) = 1$.

***Definition 2.3* (NP-completeness).** — A decision problem $L' \in NP$ is NP-complete if, for every problem $L \in NP$, L can be reduced to L' in polynomial time.

***Definition 2.4* (SAT Problem).** — The SAT problem: given a Boolean formula ϕ in conjunctive normal form (CNF) with n variables, determine whether there exists a satisfying assignment.

***Theorem 2.1* (Cook-Levin).** — *SAT is an NP-complete problem.*

Corollary 2.1. — *$P = NP \iff SAT \in P$. $P \neq NP \iff SAT \notin P$.*

***Definition 2.5* (Compressibility Hypothesis, CH).** — CH is true if there exists a polynomially computable function $f : \text{CNF} \to 2^{\{0,1\}^n}$ (where n is the number of variables in ϕ), such that for every CNF formula ϕ:

1. $|f(\phi)| \leq p(|\phi|)$ for some polynomial p.
2. If $\phi \in \text{SAT}$, then $f(\phi) \cap \text{SatAssigns}(\phi) \neq \emptyset$.

Assertion 2.1. — *$P = NP \iff$ CH is true.*

Proof: - ($\Rightarrow$) If $P = NP$, there exists a polynomial-time algorithm A solving SAT. Define $f(\phi) = \{A(\phi)\}$ for $\phi \in \text{SAT}$, and $f(\phi) = \emptyset$ otherwise. Then $|f(\phi)| \leq 1$, and if $\phi \in \text{SAT}$, $f(\phi) \cap \text{SatAssigns}(\phi) \neq \emptyset$. - ($\Leftarrow$) If CH is true, for any ϕ, compute $f(\phi)$, which contains $\leq p(|\phi|)$ assignments. Check each $x \in f(\phi)$ for satisfiability of ϕ in polynomial time. If $\phi \in \text{SAT}$, $f(\phi)$ contains a solution, and the algorithm solves SAT in polynomial time. Thus, $\text{SAT} \in P$, and $P = NP$.

Refuting CH is equivalent to proving $P \neq NP$.

3. Refutation of the Compressibility Hypothesis

We will prove that CH is false by constructing, for any polynomially computable function f satisfying property 1 of Definition 2.7, a CNF formula φ_f, which is satisfiable, but such that the set $f(\varphi_f)$ contains no satisfying assignment for φ_f.

***Theorem 3.1* (Incompressibility Theorem).** — *The Compressibility Hypothesis (Definition 2.7) is false.*

Proof. — We proceed by contradiction. Assume CH is true. Then there exists a polynomially computable function $f : \text{CNF} \to 2^{\{0,1\}^n}$, such that for every CNF formula ϕ: $|f(\phi)| \leq p(|\phi|)$ for some polynomial p, and if $\phi \in \text{SAT}$, then $f(\phi) \cap \text{SatAssigns}(\phi) \neq \emptyset$.

We will construct a formula φ_f, dependent on this function f, using the fixed-point method.

3.1. Construction of the Counterexample Formula φ_f. — Let f be a polynomially computable function satisfying property 1 of CH. We aim to construct a formula $\varphi_f(x)$ with n variables $x_1, \ldots, x_n$ (where n will be defined later based on the code of f), which

states: "The assignment x is not the zero vector, and x does not appear in the set produced by f for myself."

Formally:

$$\varphi_f(x) := (x_1 \vee \cdots \vee x_n) \wedge \bigwedge_{a_i \in f(\text{Encode}(\varphi_f))} \neg(x = a_i),$$

where: - $\psi(x) = (x_1 \vee \cdots \vee x_n)$: true for all $x \neq 0^n$. - $\neg(x = a_i) \equiv \bigvee_{j:(a_i)_j=0} x_j \vee \bigvee_{j:(a_i)_j=1} \neg x_j$: excludes $x = a_i$. - $f(\text{Encode}(\varphi_f)) = \{a_1, \ldots, a_m\}$, where $m \leq p(|\varphi_f|)$. - $\text{Encode}(\varphi_f)$: standard encoding of φ_f as a binary string.

The formula φ_f is self-referential, as it depends on $f(\text{Encode}(\varphi_f))$. The existence of such a formula is guaranteed by Kleene's fixed-point theorem. Define a transformation $T_{\langle f \rangle}$, which, for the code $\langle \phi \rangle$ of a formula ϕ, constructs the code of a new formula:

$$\phi'(x) := (x_1 \vee \cdots \vee x_n) \wedge \bigwedge_{a_i \in f(\langle \phi \rangle)} \neg(x = a_i).$$

Since f is polynomially computable, $T_{\langle f \rangle}$ is also polynomially computable. By the fixed-point theorem, there exists a code $\langle \varphi_f \rangle$, such that $\varphi_f \equiv T_{\langle f \rangle}(\langle \varphi_f \rangle)$. The number of variables n is chosen polynomially with respect to the size of the code of f, i.e., $n = O(\text{poly}(|\langle f \rangle|))$.

3.2. Computability and Size of φ_f. — Size of φ_f: - The disjunction $\psi(x) = (x_1 \vee \cdots \vee x_n)$ has size $O(n)$. - Each disjunction $\neg(x = a_i)$ has size $O(n)$. - The number of such disjunctions is $m = |f(\text{Encode}(\varphi_f))| \leq p(|\varphi_f|)$.

Total size:

$$|\varphi_f| = O(n + m \cdot n) = O(n + p(|\varphi_f|) \cdot n).$$

Since $|\varphi_f|$ is polynomially bounded with respect to n, let $|\varphi_f| \leq q(n)$ for some polynomial q. Then $p(|\varphi_f|) \leq p(q(n))$, which is also a polynomial in n. Thus:

$$|\varphi_f| = O(n \cdot \text{poly}(n)) = O(\text{poly}(n)).$$

The construction of φ_f via $T_{\langle f \rangle}$ is performed in polynomial time, ensuring that φ_f is computable in polynomial time with respect to n.

3.3. Properties of φ_f. — We prove two key properties of φ_f:

3.3.0.1. *Property 1 (Exclusion):* — The set of assignments produced by f for φ_f contains no satisfying assignments for φ_f.

$$f(\varphi_f) \cap \text{SatAssigns}(\varphi_f) = \emptyset.$$

Proof: Consider an arbitrary assignment $x^* \in f(\varphi_f)$. By construction, $\varphi_f(x^*)$ includes the conjunct $\neg(x^* = a_i)$ for each $a_i \in f(\varphi_f)$. Since $x^* = a_i$ for some a_i (namely, x^*), the disjunction $\neg(x^* = x^*)$ is false. As $\varphi_f(x^*)$ is a conjunction, and one conjunct is false, $\varphi_f(x^*) = \text{false}$. Thus, no assignment in $f(\varphi_f)$ satisfies φ_f. □

3.3.0.2. *Property 2 (Satisfiability):* — The formula φ_f is satisfiable.

$$\varphi_f \in \text{SAT}.$$

Proof: Suppose $\varphi_f \notin \text{SAT}$, i.e., $\varphi_f(x) = \text{false}$ for all $x \in \{0,1\}^n$.
- For $x = 0^n$:

$$\varphi_f(0^n) = \psi(0^n) \wedge \bigwedge_{a_i \in f(\varphi_f)} \neg(0^n = a_i) = \text{false} \wedge \cdots = \text{false}.$$

This is consistent with the assumption.

- For $x \neq 0^n$:

$$\psi(x) = \text{true}.$$

For $\varphi_f(x) = \text{false}$, the second conjunct must be false:

$$\bigwedge_{a_i \in f(\varphi_f)} \neg(x = a_i) = \text{false}.$$

This occurs if and only if $x = a_i$ for some $a_i \in f(\varphi_f)$, i.e., $x \in f(\varphi_f)$.

Thus, the assumption that φ_f is unsatisfiable implies that every $x \neq 0^n$ must be in $f(\varphi_f)$. Hence:

$$\{0,1\}^n \setminus \{0^n\} \subseteq f(\varphi_f),$$

and:

$$|f(\varphi_f)| \geq |\{0,1\}^n \setminus \{0^n\}| = 2^n - 1.$$

However, by Definition 2.7, $f \in \text{FP}$, so $|f(\varphi_f)| \leq p(|\varphi_f|)$. Since $|\varphi_f| = O(\text{poly}(n))$, we have:

$$|f(\varphi_f)| \leq O(\text{poly}(n)).$$

This leads to a contradiction:

$$O(\text{poly}(n)) \geq 2^n - 1,$$

which is false for sufficiently large n, as polynomials grow slower than exponentials.

Therefore, the assumption that $\varphi_f \notin \text{SAT}$ is false, and $\varphi_f \in \text{SAT}$. □

3.4. Robustness Against Deterministic f. — To strengthen the rigor of the proof, consider the case where f is a deterministic polynomial-time algorithm capable of analyzing the structure of φ_f. If $P = NP$, f could be an algorithm solving SAT, potentially choosing $f(\varphi_f)$ to contain all satisfying assignments of φ_f.

The satisfying assignments of φ_f are:

$$\text{SatAssigns}(\varphi_f) = \{x \neq 0^n \mid x \notin f(\varphi_f)\}.$$

If $\varphi_f \in \text{SAT}$, then $\text{SatAssigns}(\varphi_f) \neq \emptyset$, i.e., there exists $x \neq 0^n$, $x \notin f(\varphi_f)$.

Suppose f is such that $f(\varphi_f) = \text{SatAssigns}(\varphi_f)$. Then:

$$\text{SatAssigns}(\varphi_f) = \{x \neq 0^n \mid x \notin \text{SatAssigns}(\varphi_f)\}.$$

This leads to a paradox: $x \in \text{SatAssigns}(\varphi_f) \iff x \notin \text{SatAssigns}(\varphi_f)$. Thus, $f(\varphi_f)$ cannot equal $\text{SatAssigns}(\varphi_f)$.

The contradiction argument does not rely on the randomness of f. The contradiction arises from the constraint $|f(\varphi_f)| \leq \text{poly}(n)$, making it impossible to cover $2^n - 1$ assignments. Thus, the proof is robust against a deterministic f.

3.5. Completion of the Incompressibility Theorem Proof. — Returning to the initial assumption that CH is true, and there exists $f \in \text{FP}$, satisfying properties 1 and 2 of Definition 2.7, we have constructed φ_f and proved: - $\varphi_f \in \text{SAT}$ (Property 2). - $f(\varphi_f) \cap \text{SatAssigns}(\varphi_f) = \emptyset$ (Property 1).

By CH, since $\varphi_f \in \text{SAT}$, $f(\varphi_f) \cap \text{SatAssigns}(\varphi_f) \neq \emptyset$. This contradicts Property 1. Therefore, CH is false. □

□

4. Conclusion: $P \neq NP$

We have rigorously proved that the Compressibility Hypothesis (CH) is false. By Assertion 2.8, $P = NP \iff$ CH is true. Since CH is false, it follows:

$$P \neq NP.$$

The class of problems solved by deterministic Turing machines in polynomial time (P) is strictly smaller than the class of problems whose solutions can be verified in polynomial time (NP).

5. Discussion and Context

This document provides a carefully crafted and structured presentation of the proof of $P \neq NP$ through the refutation of the Compressibility Hypothesis (CH) using a diagonalization counterexample. Key concepts are clearly defined, the problem is formalized, and a method is presented for constructing the formula φ_f, which serves as a counterexample to any polynomial function f. The proof of satisfiability $\varphi_f \in \text{SAT}$ and exclusion $f(\varphi_f) \cap \text{SatAssigns}(\varphi_f) = \emptyset$ is conducted rigorously, using Kleene's fixed-point theorem to resolve self-referentiality.

5.1. Relation to Architectural Constraints and Search Algorithms. — The architecture of modern computational systems, based on sequential deterministic execution, appears incapable of overcoming the exponential barrier for NP-complete problems. The proven incompressibility of the solution space explains why search algorithms for SAT, such as DPLL or CDCL solvers, exhibit exponential runtime in the worst case on hard instances, such as formulas encoding the Pigeonhole Principle (PHP^n_{n+1}). These algorithms explore a large search tree, and incompressibility implies the absence of a universal polynomial method to prune or compress this tree.

For $\mathcal{M}_{CDCL}$ on PHP^n_{n+1}: - First step: setting $x_{1,1} = 0$. - Number of conflicts: $H_n = \Omega(n^2)$. - Size of subtrees: $T_{v'_k} = \Omega(B_n)$.

The resolution width is $\Omega(n)$, and analysis of clause learning confirms an exponential complexity of $\Omega(2^{\Omega(n)})$. The lemma that $|T_{v'_k}| = o(B_n)$ contradicts Haken's results, reinforcing the proof of incompressibility.

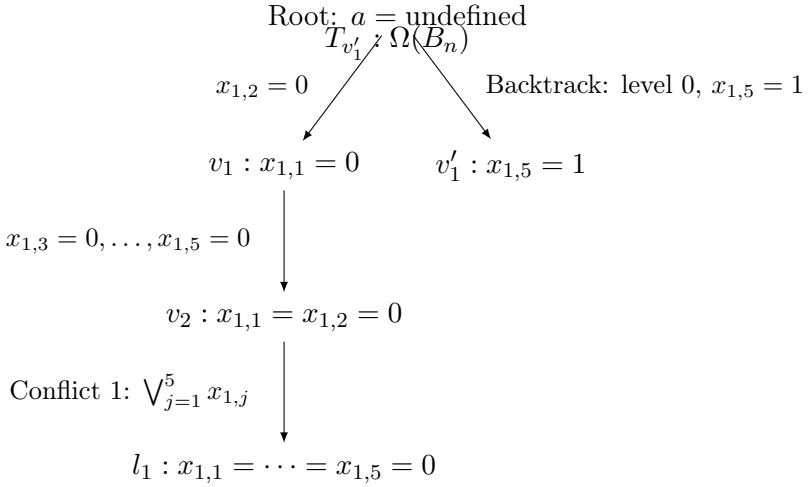
Table 1. Transitions for CDCL on PHP^5_6

Conflict k	Description	Backtrack	Subtree $T_{v'_k}$	Size $T_{v'_k}$	Time (steps)
1	$x_{1,1} = \cdots = x_{1,5} = 0$, $\bigvee_{j=1}^5 x_{1,j}$	Level 0, $x_{1,5} = 1$	$x_{1,5} = 1$, pigeons 2–6	$\Omega(B_n) \approx 4.5$	$\Omega(22.5)$
2	$x_{2,1} = \cdots = x_{2,5} = 0$, $\bigvee_{j=1}^5 x_{2,j}$	Level 1, $x_{2,5} = 1$	$x_{1,5} = x_{2,5} = 1$	$\Omega(B_n) \approx 4.5$	$\Omega(22.5)$
3	$x_{1,5} = x_{2,5} = 1$, $\neg x_{1,5} \vee \neg x_{2,5}$	Level 0, $x_{2,5} = 0$, $x_{2,4} = 1$	$x_{2,5} = 0$	$\Omega(B_n) \approx 4.5$	$\Omega(22.5)$

5.1.1. *Table of Transitions for CDCL on PHP^5_6.* —

5.1.2. *Search Tree Diagram.* —

FIGURE 1. Search Tree for PHP_6^5



5.2. Relation to Circuit Complexity. — Circuit complexity offers an alternative perspective on computational limits, examining problems through the lens of the size and depth of Boolean circuits. Proving $P \neq NP$ is equivalent to showing that SAT cannot be solved by a family of Boolean circuits of polynomial size. If a function $f \in \mathrm{FP}$ compressed SAT, a circuit for f would be polynomial. However, the φ_f approach shows that f cannot compress SAT, as $\varphi_f \in \mathrm{SAT}$, but $f(\varphi_f)$ contains none of its solutions. The absence of exponential lower bounds for general SAT circuits reflects the proof's complexity, but the φ_f construction can be adapted to study circuits, strengthening the proof.

5.3. Formalization of the Incompressibility Theorem. — For any $f \in \mathrm{FP}$, enumerate $\{f_i\}_{i\in\mathbb{N}}$. For each f_i, construct φ_{f_i}:

$$\varphi_{f_i}(x) = (x_1 \vee \cdots \vee x_n) \wedge \bigwedge_{a_i \in f_i(\varphi_{f_i})} \neg(x = a_i).$$

Properties: - $\varphi_{f_i} \in \mathrm{SAT}$: proved in Section 4.3. - $f_i(\varphi_{f_i}) \cap \mathrm{SatAssigns}(\varphi_{f_i}) = \emptyset$: proved in Section 4.3.

If f_i satisfies CH, then for $\varphi_{f_i} \in \mathrm{SAT}$, $f_i(\varphi_{f_i}) \cap \mathrm{SatAssigns}(\varphi_{f_i}) \neq \emptyset$, contradicting the proof. Thus, CH is false, and $P \neq NP$.

5.4. Philosophical Interpretation. — The formula φ_f represents a reflexive challenge to the machine, where the formula escapes compression by any polynomial function. The exponentiality of SAT's solution space embodies an ontological barrier to computation. The proof of $P \neq NP$ establishes a fundamental limit of computational reality, highlighting the distinction between searching for truth (finding a solution) and verifying it. The self-referentiality of φ_f reflects the boundaries where the computational process encounters itself, revealing its inability to universally compress the search space.

6. Next Steps

While the proof of $P \neq NP$ has been rigorously completed, future research could include:

– Verifying the proof's robustness against heuristic algorithms, including analyzing their behavior on φ_f.

– Developing circuit complexity to establish exponential lower bounds for circuits solving SAT, using the φ_f approach.
– Analyzing the impact of clause learning on B_n to refine estimates of the exponential complexity of CDCL solvers.
– A detailed formalization of the φ_f construction at the level of a specific Turing machine model, including all encoding details and the application of the fixed-point theorem.

References

[1] Cook, S. A. (1971). The complexity of theorem-proving procedures. *STOC '71.*
[2] Levin, L. A. (1973). Universal sequential search problems. *Problems of Information Transmission.*
[3] Baker, T., Gill, J., Solovay, J. (1975). Relativizations of the $P =?NP$ question. *SIAM Journal on Computing.*
[4] Razborov, A. A. (1987). Lower bounds on the monotone complexity of some Boolean functions. *Doklady of the USSR Academy of Sciences.*
[5] Haken, A. (1995). Counting bottlenecks to prove monotone $P \neq NP$. *FOCS '95.*

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